Scaling of Spectral Anisotropy in Two Dimensional Electron Magnetohydrodynamic Turbulence

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Abstract

It is demonstrated using numerical simulation that spectral anisotropy in two dimensional decaying Electron Magnetohydrodynamic turbulence varies linearly with the ratio of fluctuating to mean magnetic field. Turbulent excitation scales larger ($k\lambda_e < 1$) as well as smaller ($k\lambda_e > 1$) than electron skin depth scale length (λ_e) are shown to exhibit such linear scaling behaviour.

KEYWORDS : electron magnetohydrodynamics, turbulence, anisotropy.

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I. INTRODUCTION

Turbulence in the magnetofluids is evidently a complex phenomena. Unlike hydrodynamical fluids, plasma fluids support a variety of basic modes. The excitation of fundamental modes in plasmas may occur due to a number of processes viz, large scale magnetic field, electric field, mean gradients of various physical quantities and others. These fundamental modes often interact with the turbulent field and influence various properties of the fluid in rather subtle manner. For example, it is widely known from the knowledge of magnetohydrodynamic (MHD) turbulence that presence of a large scale magnetic field influences spectral power cascade mechanism in a complicated manner [2]. Furthermore, it has been shown that the presence of a mean (or dc) magnetic field renders anisotropy in the MHD turbulent spectrum [3]. The existence of a mean magnetic field excites fundamental MHD modes, known as Alfvén waves, in the turbulent fluid. These propagating waves in presence of mean magnetic field give rise to distinct energy cascade rates along and across large scale magnetic field, thereby leading to asymmetry in spectral transfer rates. Such asymmetric cascade is called as spectral anisotropy. However these subtleties are still a debatable issue [4–6]. On the other hand, there have been plenty of evidences of anisotropy in the laboratory plasma [7,8], solar winds [9,10], magnetohydrodynamical plasmas [3,15].

Alfvénic turbulence describes relatively large length scale and low frequency MHD plasma turbulence. Recently, there has been considerable interests in a new paradigm of plasma known as Electron Magnetohydrodynamics (EMHD), which is basically characterized by relatively high frequency and smaller length scale [11]. EMHD model is of particular importance in astrophysical plasmas, magnetic reconnection phenomena, fast plasma opening switches, anomalous diffusion of field in plasma, and others [11,12]. It has been reported in the recent works that the fundamental oscillatory modes of EMHD, whistler waves, render spectral anisotropy in two dimensional EMHD turbulence by weakening energy transfer rates along large scale magnetic field [13,14]. It appears then that the basic mechanism of generation of spectral anisotropy in both the paradigm of plasma viz, MHD and EMHD is similar.

However they differ in the strength of the interaction along large scale magnetic field, which is fairly restrictive and almost absent in the Alfvénic turbulence because of nondispersive nature of Alfvén waves. This is unlike whistler turbulence in which such interactions are weak but remain present. The effects due to presence of self-consistent large scale magnetic field such as the whistlerization phenomena, their scale dependence feature and others have also been reported in the earlier works [13,14].

In this letter we would like to report on dependence of anisotropy on the strength of an external mean dc magnetic field. Our investigations based upon numerical simulation of 2D EMHD equation reveal that anisotropy systematically varies with the strength of an external magnetic field. We further seek a relationship between anisotropy and external mean magnetic field. We also investigated effect of dissipation parameter on anisotropy. In section II, we discuss basic equations of 2D EMHD, outline our numerical scheme and present simulation results. Section III, contains discussion on the numerical results.

II. BASIC EQUATIONS & NUMERICAL RESULTS

Electron Magnetohydrodynamics (EMHD) is the single fluid description of quasi-neutral magnetized plasma phenomena which occur on relatively fast time scales as compared to the usual MHD phenomena [11]. On such time scales, only electron dynamics is important. On the other hand, ions merely provide a stationary neutralizing background against fast moving electrons and behave as scattering centers. The EMHD model has been discussed in considerable detail in some of the earlier works [11,4–6,13,14]. Here, we concentrate on the case when the magnetic field perturbations are restricted to two dimensions, e.g. the $\boldsymbol{x-y}$ plane, and $\boldsymbol{\partial/\partial z = 0}$. Further, all the three components of the perturbed magnetic field are retained. The model equations describing 2D EMHD, can then be cast into the following set of coupled nonlinear equations in terms of two field variables *viz.* $\boldsymbol{b}$ and $\boldsymbol{\psi}$.

$$\frac{\partial}{\partial t}(\psi - \nabla^2\psi) + \hat{z} \times \nabla b \cdot \nabla(\psi - \nabla^2\psi) = \mu\nabla^{2n}\psi \tag{1}$$

$$\frac{\partial}{\partial t}(b - \nabla^2 b) - \hat{z} \times \nabla b \cdot \nabla\nabla^2 b + \hat{z} \times \nabla\psi \cdot \nabla\nabla^2\psi = \mu\nabla^{2n} b \qquad (2)$$

The total magnetic field can be expressed in terms of these fields as $\vec{B} = \hat{z} \times \nabla\psi + b\hat{z}$. The two evolution equations have been written in normalized variables. The length scales are normalized by the electron skin depth $\lambda_e = c/\omega_{pe}$, magnetic field by a typical amplitude B_0 and time by the corresponding electron gyrofrequency. In Eqs. (1) & (2), the diffusion operator on the right hand side is raised to $2n$. Here n is an integer and can take $n = 1, 2, 3, \cdots$. $n = 1$ corresponds to normal diffusion, while $n = 2, 3, \cdots$ correspond to hyper- and higher order diffusion terms. The linearization of Eqs. (1) & (2) about a constant magnetic field yields dispersion relation for whistlers, the normal mode of oscillation in the EMHD frequency regime. Our objective here is to investigate the effect and influence of varying B_0 on anisotropy exhibited by Eqs. (1) & (2). For this purpose we simulate Eqs. (1) & (2) for varying strength of B_0. The constant magnetic field (B_0) is chosen along $\hat{y}$ direction. The two field EMHD equations are numerically integrated with the help of a fully de-aliased pseudospectral scheme. Periodic boundary conditions are imposed along x and y-directions. The evolution variables b, ψ are discretized in fourier space. The linear part of Eqs. (1) & (2) is integrated exactly. The nonlinear terms are evaluated in real space and then fourier transformed in k-space by going back and forth in real and k-space at each time step. The Fast Fourier Transform (FFT) routines are used to go back and forth in the real and k-space at each time integration. The time advancement is done using predictor corrector with the mid point leap frog scheme. The spatial resolution varies from 256^2 to 512^2 fourier modes. Higher order of dissipation operator in Eqs. (1) & (2) was used to restrict dissipative effects at shorter scales. The typical values of the dissipation parameter μ was chosen to lie between 1×10^{-6} to 1×10^{-7} in the various runs. The initial spectrum of the two fields b and ψ was chosen to be sufficiently isotropic and phases were taken to be random. The fourier modes of the two fields are chosen to be entirely uncorrelated to begin with. The evolution of wavenumbers along and across mean magnetic field has been respectively quantified as follow [15];

$$\langle \bar{k}_{||}(t) \rangle = \sqrt{\frac{\sum_{\vec{k}} k_{||}^2 |\Omega(\vec{k},t)|^2}{\sum_{\vec{k}} |\Omega(\vec{k},t)|^2}} = |\vec{k}| \sin\Theta \tag{3}$$

$$\langle \bar{k}_{\perp}(t) \rangle = \sqrt{\frac{\sum_{\vec{k}} k_{\perp}^2 |\Omega(\vec{k},t)|^2}{\sum_{\vec{k}} |\Omega(\vec{k},t)|^2}} = |\vec{k}| \cos\Theta \tag{4}$$

where $\vec{k} = k_\perp \hat{x} + k_{||} \hat{y}$ is total wave vector, $k_{||}$ and $k_\perp$ denote wavenumber along and across mean magnetic field B_0 respectively. $\omega = \sum_{\vec{k}} \Omega(\vec{k},t) = \nabla^2 \psi$ or $\nabla^2 b$. The sum is over all $\vec{k}$'s and $k_{||}/k_\perp = \tan\Theta$ is called as angle of anisotropy. It is clear from the definition of angle of anisotropy that it hovers around unity for an isotropic turbulent flow thereby indicating that spectral power along and across an external magnetic field is similar i.e, $k_{||} \simeq k_\perp$. The latter will not hold when turbulence is sufficiently anisotropic. This is because of asymmetry in spectral transfer which can occur as a result of disparity between evolution of $k_{||}$ and $k_\perp$ in presence of an external magnetic field. Such results within the frame work of EMHD, have been reported in the earlier work [13,14].

Next we present the numerical results. It is observed in the simulation that when the strength of an external magnetic field (B_0) is increased, turbulent spectrum becomes more anisotropic. The increasing anisotropic behaviour of the spectrum has systematic dependence on the increasing strength of B_0. This is demonstrated in Fig (1) for $k\lambda_e < 1$ case. Figure (1) represents variation of sine of anisotropic angle as a function of ratio of fluctuating to the mean magnetic field. The anisotropic angle $\Theta = \tan^{-1}(k_{||}/k_\perp) \simeq 45^\circ$ or $\sin\Theta \simeq 0.707$ represents an isotropic turbulent spectrum. Any deviation from this value would be attributed to anisotropic spectrum. We have carried out simulation for different strengths of B_0. It can be seen from Fig (1) that higher the strength of B_0 (i.e. lower $\tilde{b}/B_0$ ratio), larger is deviation from the isotropic value. This further indicates that anisotropy is more when the strength of B_0 is more. It is clear from the Fig (1) that anisotropy acquires fairly linear relationship with the strength of external magnetic field. Each point in the Fig (1) corresponds to simulation run in which turbulence has attained almost saturated anisotropic spectrum. The anisotropic angle is then averaged over a few periods of the saturated spectrum. The best possible fit to these points corresponds to a straight line (see Fig. 1), which clearly demonstrates the linear relationship between anisotropy and an external $\boldsymbol{B_0}$.

The linear scaling as demonstrated above seems to be a robust feature of 2D EMHD turbulence. Similar kind of scaling has also been observed in the regime when turbulent excitation length scales are smaller than electron skin depth i.e. $\boldsymbol{k\lambda_e > 1}$ region. This is shown in Fig. (2). Thus Figs (1) & (2) show that in EMHD turbulence the sine of anisotropy angle varies as the ratio of fluctuating to mean magnetic field, and hence the following relationship;

$$\sin\Theta \sim \frac{\tilde{\boldsymbol{b}}}{\boldsymbol{B_0}}$$

This is an interesting observation in the context of EMHD which holds for $\boldsymbol{k\lambda_e < 1}$ and $\boldsymbol{k\lambda_e > 1}$ regimes of turbulence. It is known that in the regime $\boldsymbol{k\lambda_e < 1}$, wave effects are dominant and influence spectral properties to much greater extent [13]. On the other hand, in the regime $\boldsymbol{k\lambda_e > 1}$, turbulence evidently acquires hydrodynamical features [14]. However it is noteworthy here that the trend of anisotropy varies in similar fashion in the two regimes. Furthermore, it is interesting to notice from Figs (1) & (2) that the straight line in $\boldsymbol{k\lambda_e < 1}$ case is relatively far away from the isotropic value i.e $\sin\Theta \simeq \mathbf{0.707}$ as compared to the other case (i.e. $\boldsymbol{k\lambda_e > 1}$). This further indicates that anisotropy in longer length scales (compared to $\boldsymbol{\lambda_e}$) regime is more pronounced than that in the smaller length scale case ($\boldsymbol{k\lambda_e > 1}$). In the latter case, turbulence possesses relatively weak tendency of anisotropy. This is again in conformity with the earlier investigations [13,14].

We next investigate the effect of dissipation parameter (i.e. $\boldsymbol{\mu}$) on the evolution of parallel as well as perpendicular wave numbers as defined by Eqs. (3) & (4). The simulation results are displayed in Fig. (3). In the figure, the saturated value of both the wavenumbers (i.e. $\boldsymbol{k_{\|}}$ and $\boldsymbol{k_{\perp}}$) is plotted against different values of $\boldsymbol{\mu}$. The upper and lower curves represent evolution of $\boldsymbol{k_{\perp}}$ and $\boldsymbol{k_{\|}}$ respectively. It is apparent from the simulation results of $\boldsymbol{k\lambda_e < 1}$ case that the ratio of the two wavenumbers ($\boldsymbol{k_{\perp}/k_{\|} > 1}$) decreases as $\boldsymbol{\mu}$ is increased. It further indicates that the two wave numbers are quite disparate at smaller length scales. The disparity decreases with increasing length scales. Consequently anisotropy is more prominent at shorter length scales in the turbulence for $k\lambda_e < 1$ case.

III. DISCUSSION

We now explain in terms of simple arguments the linear relationship (i.e. $\sin\Theta \propto b/B_0$) as observed in the numerical simulation of 2D EMHD Eqs. (1) & (2). We define a parameter called as anisotropic parameter, which is a ratio of nonlinear frequency to linear frequency i.e. $\Delta = \Omega_{NL}/\omega$. In the regime where the excitation scales are smaller than electron skin depth, linear dispersion relation of whistler yields $\omega \simeq k k_{\|} B_0$ (where $k_{\|}$ is wave number along mean magnetic field). The nonlinear frequency can be obtained from convective nonlinearity in the EMHD as, $\Omega_{NL} \simeq k^2 b_k$. The ratio of these two frequencies can be written as follows,

$$\frac{b}{B_0} \simeq \left(\frac{\Delta}{k}\right) k_{\|} \tag{5}$$

A close look at Eq. (5) states that spectral cascade along an external magnetic field suppresses more (i.e. $k_{\|}$ reduces), when the strength of B_0 is more. This is indeed observed in the simulation also, and is shown in Fig (4). Figure (4) displays variation of parallel wavenumber ($k_{\|}$) as a function of the ratio $\tilde{b}/B_0$. It is clear from this figure that when the strength of external magnetic field is high (i.e. low $\tilde{b}/B_0$ ratio), parallel cascade also reduces linearly. Further, from the definition of $k_{\|}$ (i.e. Eq. (3)), one can write $k_{\|}/k = \sin\Theta$. On substituting the later in Eq. (5), we can write

$$\sin\Theta \propto \frac{b}{B_0}$$

Similar scaling can be drawn in the regime $k\lambda_e > 1$. It is clear that the linear scaling of anisotropy is directly related with the strength of an external magnetic field. Therefore when the strength of B_0 is more, spectral transfer processes are mainly dominated by the linear whistler waves, and nonlinear transfer rates become slower compared to the linear one. As a result, relatively large transfer of energy takes place along parallel scales. Consequently the parallel scales appear longer than perpendicular scales. Furthermore, the discrepancy in the two scales increases as the strength of an external magnetic field ($\boldsymbol{B_0}$) is increased.

In conclusion, it is demonstrated that anisotropic turbulent spectrum in an incompressible 2D EMHD turbulence, exhibits a linear relationship with an external constant magnetic field. The mechanism appears to be very robust. Both length scale regimes (i.e. longer and shorter than electron skin depth) reveal that anisotropy linearly depends upon the ratio of fluctuating to mean magnetic field. We further expect that various other effects such as compressiblity of electron fluid, anisotropic dissipation rates, various forcing mechanism may have profound influence on the anisotropic turbulent spectrum. It is further interesting to investigate the effect of relatively higher ratio of $\boldsymbol{\tilde{b}/B_0(\gg 1)}$ on the anisotropy. Some of these issues shall be addressed in near future.

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List of Figures

- **Fig (1)** Variation of sine of anisotropic angle ($\boldsymbol{\Theta}$) as a function of ratio of fluctuating to mean magnetic field ($\tilde{\boldsymbol{b}}/\boldsymbol{B_0}$). Each point in the figure represents a time averaged over which the anisotropic turbulent spectra becomes saturated. Simulation resutls in this figure corresponds to large scale regime i.e. $\boldsymbol{k\lambda_e < 1}$. Clearly the best fit is a straight lines, indicating that anisotropic angle is directly proportional to strength of a constant magnetic field.

- **Fig (2)** Similar to Fig (1), except the turbulent excitation length scales are smaller compared to electron skin depth $\boldsymbol{k\lambda_e > 1}$. Notice here similar trend as observed in Fig (1). Comparison with Fig (1) reveals that anisotropy in this regime is less pronounced.

- **Fig (3)** Effect of dissipation parameter ($\boldsymbol{\mu}$) on parallel ($\boldsymbol{k}_{\|}$) and perpendicular ($\boldsymbol{k}_{\perp}$) wavenumbers. The effect of decreasing dissipation is to increase anisotropy. Anisotropy is more predominant at shorter length scales in $\boldsymbol{k\lambda_e < 1}$ case.

- **Fig (4)** A plot of $\boldsymbol{k}_{\|}$ Vs $\tilde{\boldsymbol{b}}/\boldsymbol{B_0}$ ratio. Higher the strength of $\boldsymbol{B_0}$ (i.e. low $\tilde{\boldsymbol{b}}/\boldsymbol{B_0}$ ratio), larger is the suppression in parallel cascade.