Example 10.1 :

Derive difference equations for the following boundary value problem

$$-2y''(x) + y(x) = \exp(-0.2x) \quad \text{(A)}$$

with the boundary condition

$$\begin{aligned} y(0) &= 1 \\ y'(10) &= -y(10) \end{aligned}$$

Assume that the grid spacing is unity.

<**solution**>

Consider the grid shown in Figure E10.1. The difference equations for $i = 1$ through 9 are as follows :

$$2(-y_{i-1} + 2y_i - y_{i+1}) + y_i \simeq \exp(-0.2i) \tag{B}$$

where $x_i = i$ is used.

For $i = 1$, the boundary condition $y_0 = y(0) = 1$ is introduced into the foregoing equations to give

$$5y_1 - 2y_2 = \exp(-0.2) + 2 \tag{C}$$

For $i = 10$, we approximate Eq.(A) first by

$$-\frac{2\left[y'(10) - y'(9.5)\right]}{1/2} + y(10) \simeq \exp(-2) \tag{D}$$

Using the central difference approximation, the term $y'(9.5)$ becomes

$$y'(9.5) \simeq \left[y(10) - y(9)\right]/1 \tag{E}$$

Introducing Eq.(E) and the right boundary condition $y'(10) = -y(10)$ into Eq.(D) yields

$$-2y_9 + 4.5y_{10} = 0.5\exp(-2) \tag{F}$$

Summarizing the difference equations obtained, we write

$$\begin{array}{rcrcrcl} 5y_1 & - & 2y_2 & & & = & \exp(-0.2) + 2 \\ -2y_{i-1} & + & 5y_i & - & 2y_{i+1} & = & \exp(-0.2x_i) \quad ,\text{for } i = 2 \text{ to } 9 \\ & - & 2y_9 & + & 4.5y_{10} & = & 0.5\exp(-2) \end{array} \tag{G}$$

where $x_j = j$ is used.

- Exact solution of Eq.(A)
  - homogeneous solution

$$-2y''(x) + y(x) = 0 \tag{1}$$

solution to Eq.(1) is

$$y_H(x) = A\exp(kx) + B\exp(-kx) \tag{2}$$

where, $k = 1/\sqrt{2}$.

  - particular solution

$$y_P(x) = \frac{1}{0.92}\exp(-0.2x) \tag{3}$$

  - general solution
    (2) and (3) gives

$$y = y_H + y_P = A\exp(x/\sqrt{2}) + B\exp(-x/\sqrt{2}) + \tfrac{1}{0.92}\exp(-0.2x) \tag{4}$$

  - boundary condition
    1. $y(0) = 1$:

$$y(0) = A + B + \frac{1}{0.92} = 1 \tag{5}$$

$$\Rightarrow A + B = -\frac{0.08}{0.92} \tag{6}$$

    2. $y'(10) = -y(10)$ & B.C.1 :

$$\Rightarrow \left(\begin{array}{lcl} A & = & 8.656 \times 10^{-5} \\ B & = & 8.687 \times 10^{-2} \end{array} \right. \tag{7}$$

```
In[1]:= y[x_] = y[x] /. Simplify[DSolve[-2 y''[x] + y[x] == Exp[-2 / 10 x], y[x], x]][[1]]
        y1[x_] = D[y[x], x] ;
        y2[x_] = D[y1[x], x] ;
        BC1 = y[0] == 1;
        BC2 = y2[10] == -y[10];
        Solve[{BC1, BC2}, {C[1], C[2]}]
        N[%]
```

Out[1]= $\frac{25\, e^{-x/5}}{23} + e^{\frac{x}{\sqrt{2}}} C[1] + e^{-\frac{x}{\sqrt{2}}} C[2]$

Out[6]= $\left\{\left\{C[1] \to -\frac{2\left(-3\, e^{2} + 26\, e^{5\sqrt{2}}\right)}{69\, e^{2}\left(-1 + e^{10\sqrt{2}}\right)},\ C[2] \to -\frac{2\, e^{-2+5\sqrt{2}}\left(-26 + 3\, e^{2+5\sqrt{2}}\right)}{69\left(-1 + e^{10\sqrt{2}}\right)}\right\}\right\}$

Out[7]= {{C[1] → -0.0000865616, C[2] → -0.08687}}

Figure 1: Mathematica code to solve Eq.(A) with BCs

- Finite difference method
  - Difference equations (in matrix form)

$$\begin{bmatrix} B_1 & C_1 & & & & & \\ A_2 & B_2 & C_2 & & & & \\ & A_3 & B_3 & C_3 & & & \\ & & & \ddots & & & \\ & & & A_i & B_i & C_i & \\ & & & & & \ddots & \\ & & & & & A_N & B_N \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \phi_i \\ \vdots \\ \phi_N \end{bmatrix} = \begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ \vdots \\ D_i \\ \vdots \\ D_N \end{bmatrix} \tag{8}$$

- Gauss elimination algorithm to solve tridiagonal equation
  - Gauss elimination

$$\left(\begin{array}{ccccccc} B_1 & C_1 & & & & & \\ 0 & (B_2' = B_2 - C_1\frac{A_2}{B_1}) & C_2 & & & & \\ & A_3 & B_3 & C_3 & & & \\ & & & \ddots & & & \\ & & & A_i & B_i & C_i & \\ & & & & & \ddots & \\ & & & & & A_N & B_N \end{array}\right) \left| \left(\begin{array}{c} D_1 \\ D_2' = D_2 - D_1\frac{A_2}{B_1} \\ D_3 \\ \vdots \\ D_i \\ \vdots \\ D_N \end{array}\right)\right.$$

$$\left(\begin{array}{ccccccc} B_1 & C_1 & & & & & \\ & B_2' & C_2 & & & & \\ & 0 & (B_3' = B_3 - C_2\frac{A_3}{B_2'}) & C_3 & & & \\ & & & \ddots & & & \\ & & & A_i & B_i & C_i & \\ & & & & & \ddots & \\ & & & & & A_N & B_N \end{array}\right) \left| \left(\begin{array}{c} D_1 \\ D_2' \\ D_3' = D_3 - D_2'\frac{A_3}{B_2'} \\ \vdots \\ D_i \\ \vdots \\ D_N \end{array}\right)\right.$$

$$\left(\begin{array}{cccccc} B_1 & C_1 & & & & \\ 0 & B'_2 & C_2 & & & \\ & 0 & B'_3 & C_3 & & \\ & & & \ddots & & \\ & & & 0 & (B'_i = B_i - C_{i-1}\frac{A_i}{B'_{i-1}}) & C_i \\ & & & & & \ddots \\ & & & & & A_N \quad B_N \end{array}\right) \left\| \left(\begin{array}{c} D_1 \\ D'_2 \\ D'_3 \\ \vdots \\ D'_i = D_i - D'_{i-1}\frac{A_i}{B'_{i-1}} \\ \vdots \\ D_N \end{array}\right)\right.$$

$$\left(\begin{array}{ccccccc} B_1 & C_1 & & & & & \\ & B'_2 & C_2 & & & & \\ & & \ddots & & & & \\ & & & B'_i & C_i & & \\ & & & & \ddots & & \\ & & & & & B'_{N-1} & C_{N-1} \\ & & & & & A_N & B_N \end{array}\right) \left\| \left(\begin{array}{c} D'_1 \\ D'_2 \\ \vdots \\ D'_i \\ \vdots \\ D'_{N-1} \\ D_N \end{array}\right)\right.$$

$$\left(\begin{array}{ccccccc} B_1 & C_1 & & & & & \\ & B'_2 & C_2 & & & & \\ & & \ddots & & & & \\ & & & B'_i & C_i & & \\ & & & & \ddots & & \\ & & & & & B'_{N-1} & C_{N-1} \\ & & & & & 0 & (B'_N = B_N - C_{N-1}\frac{A_N}{B'_{N-1}}) \end{array}\right) \left\| \left(\begin{array}{c} D'_1 \\ D'_2 \\ \vdots \\ D'_i \\ \vdots \\ D'_{N-1} \\ D'_N = D_N - D'_{N-1}\frac{A_N}{B'_{N-1}} \end{array}\right)\right.$$

Result :

$$\left(\begin{array}{ccccccc} B_1 & C_1 & & & & & \\ & B'_2 & C_2 & & & & \\ & & \ddots & & & & \\ & & & B'_i & C_i & & \\ & & & & \ddots & & \\ & & & & & B'_{N-1} & C_{N-1} \\ & & & & & & B'_N \end{array}\right) \left(\begin{array}{c} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_i \\ \vdots \\ \phi_{N-1} \\ \phi_N \end{array}\right) \left(\begin{array}{c} D'_1 \\ D'_2 \\ \vdots \\ D'_i \\ \vdots \\ D'_{N-1} \\ D'_N \end{array}\right)$$

And use the method of backward-substitution.

- algorithm

1. Initialize the two new variables :

$$B'_1 = B_1 \text{ and } D'_1 = D_1 \tag{9}$$

2. Recurrently calculate the following equations in increasing order of i until $i = N$ is reached :

$$\begin{aligned} R &= A_i / B'_{i-1} \\ B'_i &= B_i - R C_{i-1} \\ D'_i &= D_i - R D'_{i-1} \quad \text{for } i = 2, 3, \ldots, N. \end{aligned} \tag{10}$$

3. Calculate the solution for the last unknown by

$$\phi_N = D'_N / B'_N \tag{11}$$

4. Calculate the following equation in decreasing order of i (backward-substitution)

$$\phi_i = (D'_i - C_i \phi_{i+1}) / B'_i, \quad i = N-1, N-2, \ldots, 1 \tag{12}$$

- Source code

```
C------CSL/F10-1.FOR     LINEAR BOUNDARY VALUE PROBLEM
      REAL A(80),B(80),C(80),D(80)

      N=10
      H=10.0/N
      DO 10 I=1, N
         X=I*H
         A(I)=-2
         B(I)=5
         C(I)=-2
         D(I)=EXP(-0.2*X)
10    CONTINUE
      D(1)=D(1)+2
      D(N)=D(N)*0.5
      B(N)=4.5
      CALL TRDG(A,B,C,D,N)
      PRINT*,'  x                    y'
      PRINT 901, 0.0, 1.0
      DO 20 I=1, N
         X=I*H
         PRINT 901, X,D(I)
20    CONTINUE
901   FORMAT(F6.3,5X,F15.6)
      STOP
      END
C*********************************
      SUBROUTINE TRDG(A,B,C,D,N)
      REAL A(80),B(80),C(80),D(80)
      DO 110 I=2, N
         R=A(I)/B(I-1)
         B(I)=B(I)-R*C(I-1)
         D(I)=D(I)-R*D(I-1)
110   CONTINUE
      D(N)=D(N)/B(N)
      DO 120 I=N-1, 1 ,-1
         D(I)=(D(I)-C(I)*D(I+1))/B(I)
120   CONTINUE
      RETURN
      END
C*********************************
```

- Result